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Optimal Time Re-Entry of Vehicles by Asymptotic Matching

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Asymptotic analysis has been widely used to study the change of vehicle trajectories. We have developed such an approach for a vehicle in extra-atmospheric flight. It is supposed to release an object in minimum time in such a way that the object attains, after a ballistic flight, a given point on the surface of the Earth; the initial time depends on the mission and can be, for example, the ignition time of the vehicle propellers. By this asymptotic analysis, we have derived a very fast analytical code of control in minimum time that could be used for onboard implementation. We have tested this code on different cases and compared the results with those obtained with direct numerical optimization. The results are very good when the asymptotic analysis is justified and still acceptable otherwise. This work can be applied to other types of trajectory modification problems and confirms the interest of the asymptotic approach.

I. Introduction

WE consider the following basic problem: a vehicle in extra-atmospheric flight, propelled by n thrusters, must attain in a minimum time T a point (in the phase space of positions and velocities) where it must release an object without any relative velocity. Therefore, at launching time the object and the center of gravity of the vehicle have identical states. The object is launched at the velocity of the center of gravity of the vehicle and must attain a given target on the surface of the Earth. Once it has been launched, the object follows a free fall trajectory and, therefore, describes a Keplerian ellipse. The vehicles propellers are still active after the launching; then the object and the vehicle do not follow the same trajectory. The vehicle may have to perform other launchings on other targets, so that a complete mission may consist of a series of such basic launchings.

Let us consider the simplified problem of commanding the center of gravity, assuming that the thrust can take any direction in space. Then, if the total duration of the thrust is small with respect to the orbital period of the vehicle around the Earth, the solution for the control can be correctly approximated by assuming a constant thrust. The attitude of the vehicle that is controlled by some of the thrusters determines the direction of the thrust. But as the time for a half-turn around any of the vehicle axes is very small compared to the thrust duration, the ratio of these two times gives another small parameter. This provides a typical situation of singular pertubation. The angles defining the attitude of the vehicle are fast variables; these variables present very fast variations near the initial point. That is to say, an initial rotational boundary layer exists. Here, the terminal conditions are such that no such fast variation appears at the terminal point. This analysis has enabled us to produce a very fast semianalytical method suitable for onboard implementation. To test the accuracy of the asymptotic analysis, we have compared the numerical results obtained using this method with those derived using a collocation method. We find that the numerical implementation of the asymptotic analysis is not only much faster than conventional methods but is also more robust, as it explicitly takes into account the asymptotic nature of the problem. The collocation method, although giving slightly more optimal solutions, needed to be tuned appropriately by placing more points in the regions of fast variations before it could work.

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II. Problem Formulation

A. Mechanical Equations and Notations

The n thrusters with mass flow rate $q_i(t)$ and gas relative-velocity V_i are placed at points M_i, $i = 1, \ldots, n$; G is the center of gravity of the vehicle (see Fig. 1). The initial conditions are such that we can neglect the effect of the atmosphere. The position of the center

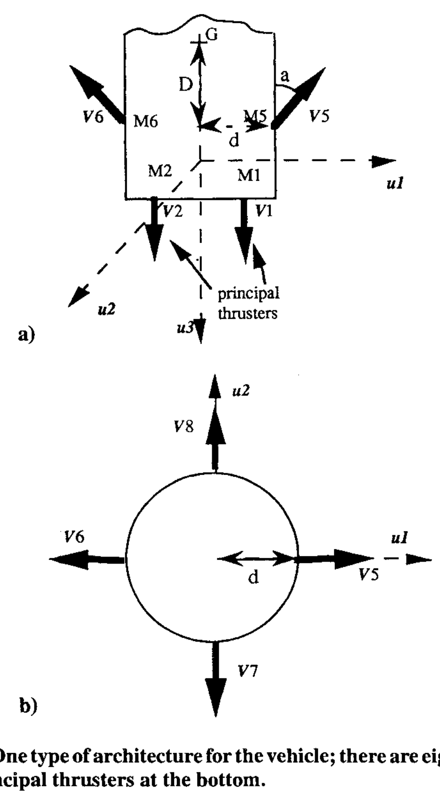

Fig. 1 One type of architecture for the vehicle; there are eight thrusters, four principal thrusters at the bottom.

of inertia is given in the Galilean frame of reference $\{O, e_1, e_2, e_3\}$ by

$$x = \sum_{i=1,3} x_i e_i$$

In this frame the Earth is supposed to be fixed; the center of the Earth is O. The coordinates for the attitude of the vehicle are the Euler angles $\{\alpha, \beta, \gamma\}$, which in turn define the principal frame of reference $\{G, u_1, u_2, u_3\}$. The moments of inertia are denoted by I_i and we suppose that $I_1 = I_2 = I$. The angular velocity is given by

$$\omega = \sum_{i=1,3} \omega_i u_i$$

Let us denote the force by F and the torque exerted by the thrusters by C, with

$$F = -\sum_{j=1,n} q_j V_j = \sum_{i=1,3} f_i(\alpha, \beta, \gamma; q_1, \ldots, q_n) e_i \tag{1a}$$

$$C = \sum_{j=1,n} q_j V_j \times GM_j = \sum_{i=1,3} C_i(\alpha, \beta, \gamma; q_1, \ldots, q_n) u_i \tag{1b}$$

The coordinates f_i and C_i of the force F and the torque C implicitly depend on the direction of the relative velocities of the gas for each thruster. As the architecture of the vehicle is given, we do not write this dependency. To fix the notation, let us write the mechanical equations of the system. The conservation of momentum then reads

$$m\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} = -\frac{km}{r^3} x_i + f_i(\alpha, \beta, \gamma; q_1, \ldots, q_n) \tag{2}$$

where the mass m of the vehicle is a function of time and k is the gravitational constant time of the Earth mass. The conservation of angular momentum is given by

$$\begin{gathered} I\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + (I_3 - I)\omega_2\omega_3 = C_1 \\ I\frac{\mathrm{d}\omega_2}{\mathrm{d}t} + (I - I_3)\omega_1\omega_3 = C_2, \qquad I_3\frac{\mathrm{d}\omega_3}{\mathrm{d}t} = C_3 \end{gathered} \tag{3}$$

The components of the force and of the torque are given by Eq. (1). The initial conditions are

$$\begin{gathered} x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0 \\ \alpha(0) = \beta(0) = \gamma(0) = 0, \qquad \omega(0) = 0 \end{gathered} \tag{4}$$

B. Terminal Condition

The released object is taken to be a single point P, its three-dimensional position being denoted by z. On a ballistic trajectory the angular momentum $L(t)$, where

$$L(t) = z \times \frac{\mathrm{d}z}{\mathrm{d}t} \tag{5}$$

is conserved, implying that the trajectory of P is imbedded in a plane orthogonal to L. Let A be the point to be attained on the Earth, whose position is defined by the vector r_A; a necessary condition for A to be reached is that the plane of the trajectory of P contains A, that is,

$$L(T) \cdot r_A = 0 \tag{6}$$

In the plane of the trajectory of P, we can use polar coordinates ρ, θ (see Fig. 2). The axes are chosen such that $\theta(T) = 0$. In these coordinates A is the point (r_T, θ_A). The equation of the trajectory of P is

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}t^2} - \rho\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = -\frac{k}{\rho^2} \tag{7}$$

which admits the usual solution

$$1/\rho = D\cos\theta + E\sin\theta + k/L^2 \tag{8}$$

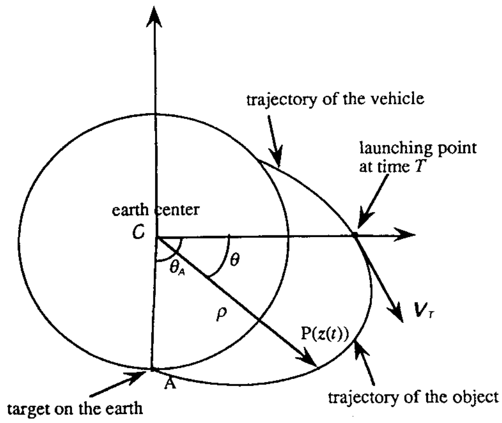

Fig. 2 System of coordinates for the object trajectory.

where L is the modulus of the vector L. The constants D and E can be expressed using the velocity and the position at the launching point, and we obtain the relation

$$\frac{1}{r_T} - \frac{k}{L^2} = \left[\frac{1}{\rho(T)} - \frac{k}{L^2}\right]\cos\theta_A - \frac{\dot{\rho}(T)}{L}\sin\theta_A \tag{9}$$

where $\dot{\rho}$ denotes the derivative of ρ with respect to time. We impose $\omega(T) = 0$, so that the initial velocity of P will be equal to the velocity of the center of gravity of the vehicle at the launching time T,

$$x(T) = z(T) \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(T) = \frac{\mathrm{d}z}{\mathrm{d}t}(T)$$

Thus, relations (6) and (9) are expressed only in terms of the terminal velocity and position of the vehicle, and we set

$$G_1[x(T), \dot{x}(T)] = L(T) \cdot r_A = 0 \tag{10a}$$

$$G_2[x(T), \dot{x}(T)] = \frac{1}{r_T} - \frac{k}{L^2} - \left[\frac{1}{\rho(T)} - \frac{k}{L^2}\right]\cos\theta_A + \frac{\dot{\rho}(T)}{L}\sin\theta_A = 0 \tag{10b}$$

As the object is launched with the velocity of the center of gravity we must add the relation

$$\omega(T) = 0 \tag{10c}$$

Therefore, the problem can be stated as follows.

Find the control $u = \{q_1(t), \ldots, q_n(t)\}$, such that the point $(x; \alpha, \beta, \gamma)$ satisfying the system of differential equations (2) and (3) attains a point satisfying the terminal conditions (10) in minimum time T. We will also suppose that the control variables satisfy the constraints

$$q_i(t) \geq 0 \qquad \sum_i q_i(t) = Q \tag{11}$$

the total mass flow rate Q being constant.

C. Reduction of the Equations

We attempt an asymptotic analysis by introducing two characteristic times: the rotation time T_2 and the foward thrust time T_1. The asymptotic analysis that will follow relies on the ratio T_1/T_2 being small. The thrust time depends on the point to be reached, but we can obtain an independent estimate for the rotation time. Let us suppose that the component C_3 of the torque is zero, which is often the case. Then no self-rotation can occur. The largest rotation that the vehicle needs to do is a half-turn around the axis 1 or 2. We can evaluate the corresponding minimum rotation time T_1. The system of equations (3) reduces to

$$I\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = C_1, \qquad I\frac{\mathrm{d}\omega_2}{\mathrm{d}t} = C_2, \qquad I_3\frac{\mathrm{d}\omega_3}{\mathrm{d}t} = 0 \tag{12}$$

the angular velocity, as a function of the Euler angles (α, β) is given by

$$\omega = \dot{\beta} \boldsymbol{u}_1 + \dot{\alpha} \sin\beta \boldsymbol{u}_2 + \dot{\alpha} \cos\beta \boldsymbol{u}_3 \tag{13}$$

the torques C_1 and C_2 are obviously related to the thrust produced by the lateral thrusters. Thus, Eq. (11) implies that C_1 and C_2 have an upper bound C,

$$(|C_1|, |C_2|) \leq C \tag{14}$$

The boundary conditions are

$$\omega(0) = \omega(T_1) = 0 \tag{15}$$

Equation (12) then gives $\mathrm{d}/\mathrm{d}t(\dot{\alpha} \cos\beta) = 0$; therefore, we can choose $\alpha = \text{const}$, and the problem (12) reduces to

$$I\ddot{\beta} = C_1, \qquad \beta(0) = 0, \qquad \beta(T_1) = \pi \qquad \dot{\beta}(0) = \dot{\beta}(T_1) = 0 \tag{16}$$

The optimal control problem is linear in the command, so that by Pontryagin's maximum principle the command must be saturated. This means, for example, that we accelerate during $T_1/2$ and decelerate during $T_1/2$ with the maximum torque.[1] We obtain for the minimum time of this approximate problem

$$T_{\min} = 2\sqrt{\pi I / C} \tag{17}$$

This time is of the order of a few seconds. We choose as reference time for the rotation

$$T_1 = \sqrt{I/C} \tag{18}$$

Let us now nondimensionalize the problem. We consider the reduced time

$$\bar{t} = t/T_2 \tag{19}$$

We set

$$\bar{k} = T_2^2 k, \qquad \bar{f}_i = f_i T_2^2 / m, \qquad C = C\bar{C} \tag{20}$$

and the problem becomes

$$\frac{\mathrm{d}^2 x_i}{\mathrm{d}\bar{t}^2} = -\frac{\bar{k}}{r^3} x_i + \bar{f}_i(\alpha, \beta, \gamma; \boldsymbol{u}), \qquad \left(\frac{T_1}{T_2}\right)^2 \frac{\mathrm{d}^2\beta}{\mathrm{d}\bar{t}^2} = \bar{C}_1(\boldsymbol{u}) \qquad \left(\frac{T_1}{T_2}\right)^2 \frac{\mathrm{d}(\dot{\alpha}\sin\beta)}{\mathrm{d}\bar{t}} = \bar{C}_2(\boldsymbol{u}) \tag{21}$$

where $\boldsymbol{u}$ is the control defined in the problem (11), the parameter $(T_1/T_2)^2 = \varepsilon$ can be small, typically, of order of magnitude 10^{-2}. This optimal control problem presents a small parameter and can be handled by singular perturbation theory.

III. Singular Perturbation Approach

A. Outer and Inner Problems

The problem just considered falls into the general frame of a singularly perturbed optimal control problem; see Calise[2–4] or Kotokovic et al.[5] In abstract form the state vector can be written as $(\boldsymbol{x}, \boldsymbol{y})$ and it satisfies the differential system

$$\dot{\boldsymbol{x}} = f_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \varepsilon, t) \qquad \boldsymbol{x} \in R^{n_1} \tag{22a}$$

$$\varepsilon\dot{\boldsymbol{y}} = f_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \varepsilon, t) \qquad \boldsymbol{y} \in R^{n_2} \tag{22b}$$

where ε is a small parameter, $\boldsymbol{x}$ is the slow variable and $\boldsymbol{y}$ is the fast variable. The solution of Eqs. (22) must satisfy the initial conditions

$$\boldsymbol{x}(\varepsilon, 0) = \boldsymbol{x}_0(\varepsilon) \tag{23a}$$

$$\boldsymbol{y}(\varepsilon, 0) = \boldsymbol{y}_0(\varepsilon) \tag{23b}$$

For the sake of simplicity we will assume no final boundary conditions. And the control must minimize the cost function

$$J = \int_0^T f_0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \varepsilon, t)\,\mathrm{d}t \tag{24}$$

If H is the Hamiltonian, we can look for a series solution in $\varepsilon, \boldsymbol{x} = \boldsymbol{x}^0(t) + \varepsilon\boldsymbol{x}^1(t) + \cdots, \boldsymbol{y} = \boldsymbol{y}^0(t) + \varepsilon\boldsymbol{y}^1(t) + \cdots, \boldsymbol{u} = \boldsymbol{u}^0(t) + \varepsilon\boldsymbol{u}^1(t) + \cdots$, verifying Eqs. (22a) and (22b) and (23a) and (23b) and $\nabla_{\boldsymbol{u}} H = 0$. The first terms of the development must be a solution of

$$\dot{\boldsymbol{x}}^0 = f_1(\boldsymbol{x}^0, \boldsymbol{y}^0, \boldsymbol{u}^0, 0, t) \qquad \boldsymbol{x} \in R^{n_1} \tag{25a}$$

$$0 = f_2(\boldsymbol{x}^0, \boldsymbol{y}^0, \boldsymbol{u}^0, 0, t) \qquad \boldsymbol{y} \in R^{n_2} \tag{25b}$$

and

$$\boldsymbol{x}^0(0) = \boldsymbol{x}_0^0 \tag{26a}$$

$$\boldsymbol{y}^0(0) = \boldsymbol{y}_0^0 \tag{26b}$$

Relation (25b) defines $\boldsymbol{y}^0$ as a function of $\boldsymbol{x}^0$ and t, so that we can express this problem in terms of the independent variable $\boldsymbol{x}^0$ alone. But if we take into account the boundary condition (26b) in Eq. (25b), we should have $0 = f_2[\boldsymbol{x}_0^0, \boldsymbol{y}_0^0, \boldsymbol{u}^0(0), 0, 0]$; this relation has no reason to be fulfilled. We, therefore, drop the initial conditions (26), and solve the so-called outer problem with initial conditions as parameters. These parameters will be used later for the matching. The fast variable will have to vary rapidly from its initial value to a value satisfying Eq. (25b). To solve this inner problem, we dilate the time near 0 setting $t = \varepsilon\tau$, and $\tilde{\boldsymbol{x}}(\tau) = \boldsymbol{x}(t), \tilde{\boldsymbol{y}}(\tau) = \boldsymbol{y}(t), \tilde{\boldsymbol{u}}(\tau) = \boldsymbol{u}(t)$. The differential system to be solved is now

$$\dot{\tilde{\boldsymbol{x}}} = \varepsilon f_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{u}}, \varepsilon, \varepsilon\tau) \qquad \boldsymbol{x} \in R^{n_1} \tag{27a}$$

$$\dot{\tilde{\boldsymbol{y}}} = f_2(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{u}}, \varepsilon, \varepsilon\tau) \qquad \boldsymbol{y} \in R^{n_2} \tag{27b}$$

In Eqs. (27) the derivatives are taken with respect to τ. The procedure is the same: we look for a series solution as previously, but now the boundary conditions are taken into account. The parameters of the outer solution are determined using the matching condition, that is, to first order,

$$\lim_{\tau\to\infty} [\tilde{\boldsymbol{x}}^0(\tau), \tilde{\boldsymbol{y}}^0(\tau), \tilde{\boldsymbol{u}}^0(\tau)] = \lim_{t\to 0} [\boldsymbol{x}^0(t), \boldsymbol{y}^0(t), \boldsymbol{u}^0(t)] \tag{28}$$

In our case here, the fast variables are the Euler angles, and the slow variables are the coordinates of the center of gravity. We could determine the matching analytically, but we will proceed numerically. Initial conditions of the outer problem will be obtained as terminal conditions for the inner problem. For the sake of consistency, because of Eq. (10c), we must impose

$$\lim_{\tau\to+\infty} \omega = 0 \qquad \text{that is} \qquad \lim_{\tau\to\infty} \dot{\alpha} = \lim_{\tau\to\infty} \dot{\beta} = 0 \tag{29}$$

Because of Eq. (29) the terminal condition (10c) is automatically verified, so that no terminal boundary layer appears.

We are led to the following procedure where one loop of the algorithm is as follows:

1) Look first for a good direction of thrust that will lead the center of gravity of the vehicle to the launching point in minimum time: here we solve the outer problem. The initial conditions are supposed to be given. The unknown here is the force $\boldsymbol{F}$ that must be applied to the center of gravity and the minimal time T_f^n. This step can be formalized in the following way.

For $n > 0$ find the control $\boldsymbol{F}$, defined in Eq. (1), that minimizes the final time T_f^n with

$$\frac{\mathrm{d}^2\boldsymbol{x}^n}{\mathrm{d}t^2} = -K\boldsymbol{x}^n + \boldsymbol{F} \tag{30a}$$

and

$$\left.\begin{matrix} \boldsymbol{x}(T_f^n) \\ \dot{\boldsymbol{x}}(T_f^n) \end{matrix}\right\} \text{belong to terminal conditions defined by Eqs. (10a) and (10b)} \tag{30b}$$

and we consider the initial conditions

$$\left.\begin{array}{l} x^n\left(T_i^n\right)\text{given} \\ \dot{x}^n\left(T_i^n\right)\text{given}\end{array}\right\} \tag{30c}$$

If the initial time T_i^n and the initial conditions $x^n(T_i^n)$ and $\dot{x}^n(T_i^n)$ are given, this problem has a unique solution denoted (T_f^n, F^n); we will see in the next paragraph that the force may be taken as constant up to a good approximation, meaning that the thrust will be constant throughout the outer problem. Given the solution, we can solve the vectorial equation $F^n = F^n(\alpha, \beta)$ with respect to the Euler angles (α, β), the solution of which will be denoted by (α_i^n, β_i^n). We begin with

$$T_i^0 = 0, \qquad x_i^{-\frac{1}{2}} = x(0), \qquad \dot{x}_i^{-\frac{1}{2}} = \dot{x}(0)$$

2) Once the direction of the thrust has been determined, we must determine how to rotate the vehicle in the direction previously obtained in such a way that the principal thruster is placed in the direction of the optimal thrust. The unknowns are the torque to apply and the minimal time T_i^n to rotate. This step can be formalized in the following way.

For $n \geq 0$ find $\min T_i^n$ such that

$$\frac{\mathrm{d}(\dot{\alpha}^{n+1}\cos\beta^{n+1})}{\mathrm{d}t} = C_1^{n+\frac{1}{2}} \qquad \frac{\mathrm{d}\dot{\beta}^{n+1}}{\mathrm{d}t} = C_2^{n+\frac{1}{2}} \tag{31a}$$

with the boundary conditions

$$\begin{array}{ll} \alpha^{n+1}(0) = \alpha_0; & \dot{\alpha}^{n+1}(0) = 0 \\ \beta^{n+1}(0) = \beta_0; & \dot{\beta}^{n+1}(0) = 0 \\ \alpha^{n+1}\left(T_i^{n+1}\right) = \alpha_i^n; & \dot{\alpha}^{n+1}\left(T_i^{n+1}\right) = 0 \\ \beta^{n+1}\left(T_i^{n+1}\right) = \beta_i^n; & \dot{\beta}^{n+1}\left(T_i^{n+1}\right) = 0 \end{array} \tag{31b}$$

The solution of the problem (31) gives T_i^{n+1} and $C^{n+1/2}$; from knowledge of the torque we will see how to calculate the mass flow rate from the thrusters during the time interval $[0, T_i^{n+1}]$.

3) During the rotation of the vehicle, the position of the center of gravity will have changed, and so we must determine the variation of this position. This step can be formalized in the following way.

During the time interval $[0, T_i^{n+1}]$ solve the differential equation

$$\frac{\mathrm{d}^2 x^{n+\frac{1}{2}}}{\mathrm{d}t^2} = -Kx^{n+\frac{1}{2}} + F\left(\alpha_i^n, \beta_i^n\right) \tag{32a}$$

with initial conditions

$$x^{n+\frac{1}{2}}(0) = x_0, \qquad \dot{x}^{n+\frac{1}{2}}(0) = \dot{x}_0 \tag{32b}$$

Steps 2 and 3 represent the inner problem. The solution of the problem (32) gives $x(T_i^{n+1})$ and $\dot{x}^n(T_i^{n+1})$, these being new initial conditions for the outer problem. One loop of the algorithm consists of two phases, which is why we have introduced the index $n + \frac{1}{2}$. We can, therefore, increase n by one and go back to problem 1 if the stopping criterion is not satisfied. This criterion depends on the difference of two successive terminal times. We now have to see how to explicitly obtain the control.

B. Semianalytical Approximation of the Control

1. Determination of the Control for the Outer Problem

Let us introduce the following notations:

$$\omega^2 = \frac{k}{r_0^3}, \qquad \sum_i q_i(t) V_i = Q V_e u \tag{33}$$

the vector u is an unitary vector; ω can be interpreted as a frequency, Q is the total mass flow rate defined in Eq. (11), and usually u is parallel to the axis u_3 of the vehicle.

We approximate the solution of the outer problem by linearizing the nonlinear gravitational term $(k/|x|^3)x$ in $(k/|x(0)|^3)x$. We set $X = (X_1, X_2) = (x, \dot{x})$ and using the variables defined in Eqs. (33) the system (30a) becomes

$$\frac{\mathrm{d}X}{\mathrm{d}t} = A \cdot X + B \tag{34a}$$

with

$$A = \begin{bmatrix} 0 & \omega I \\ -\omega I & 0 \end{bmatrix}, \qquad B = \left[0, \ -\frac{QV_e}{m(t)}u\right] \tag{34b}$$

where I is the 3×3 unit matrix. The state vector must satisfy the two terminal constraints at the final time. If $p = (p_1, p_2)$ is the adjoint state, the Hamiltonian is

$$H = p \cdot (A \cdot X + B) + p_0 \tag{34c}$$

and the adjoint state solution of the adjoint system, with the six-dimensional vector $\lambda = (\lambda_1, \lambda_2)$ as initial values, is

$$p = e^{-\omega t A} \cdot \lambda = (\cos\omega t I + \sin\omega t A) \cdot \lambda \tag{35}$$

We now inject Eq. (35) in the Hamiltonian (34c), and by the maximum principle the term to maximize is the one depending on the control, that is,

$$H_r = -[QV_e/m(t)](\sin\omega t\lambda_1 - \cos\omega t\lambda_2) \cdot u \tag{36}$$

Here the control vector u belongs to a sphere, and it should be in the direction of the vector $h = \sin\omega t\lambda_1 - \cos\omega t\lambda_2$. But h varies on an ellipse if (λ_1, λ_2) are independent vectors, or on a line segment if they are not independent. The period of h is about 5000 s; this period is generally much greater than the total time of the thrust considered here. But u has the same period, so that it varies only slightly during the time of the thrust. It is then legitimate to look for an approximate suboptimal solution in the domain of constant thrust. The control vector represents two unknowns and the terminal time one unknown. We have, then, three unknowns, and, in fact, three equations: the problem is therefore solvable.

We attempt to solve the minimum time problem in the domain of constant control variables. For the sake of simplicity we suppose that the mass is a linear function of time given by $m(t) = m_0 - Qt$. The solution of Eq. (34a) with u considered as a constant is

$$\begin{aligned} x(t) &= x_0\cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t - QV_e u\left(\int_0^t \frac{\sin[\omega(s-t)]}{m(s)}\,\mathrm{d}s\right) \\ \dot{x}(t) &= -\omega x_0\sin\omega t + \dot{x}_0\cos\omega t - QV_e u\left(\int_0^t \frac{\cos[\omega(s-t)]}{m(s)}\,\mathrm{d}s\right) \end{aligned} \tag{37}$$

We want to satisfy the terminal conditions (10a) and (10b), and so we inject Eqs. (37) into the condition (10a) and after some algebra obtain the relation

$$(x_0 \times \dot{x}_0) \cdot r_A = QV_e l(t) \cdot u \tag{38a}$$

with

$$l(t) = \left(\omega x_0 \int_0^t \frac{\cos\omega s}{m(s)}\,\mathrm{d}s + \dot{x}_0 \int_0^t \frac{\sin\omega s}{m(s)}\,\mathrm{d}s\right) \times r_A \tag{38b}$$

The final time and the control u must satisfy Eq. (38a). This equation has no solution if

$$|l(t)| < \frac{|(x_0 \times \dot{x}_0) \cdot r_A|}{QV_e} \tag{39}$$

In the range of value of the parameters, $|l(t)|$ is an increasing function of time. Then either there is no solution to the problem, or there is a minimum time $\tau_{\min}$ solution of

$$|l(t)| = \frac{|(x_0 \times \dot{x}_0) \cdot r_A|}{QV_e} \tag{40}$$

For $t > \tau_{\min}$ the set of all controls that verify Eq. (38a) is a cone parametrized by the angle γ, $\gamma \in [0, 2\pi]$, such that

$$u = k\frac{l(t)}{|l(t)|} + \sqrt{1-k^2}\left\{\cos\gamma r_A + \sin\gamma\frac{l(t)\times r_A}{|l(t)|}\right\} \tag{41a}$$

with

$$k = \frac{(x_0 \times \dot{x}_0)\cdot r_A}{QV_e|l(t)|} \tag{41b}$$

We can now use Eq. (41a) for solving numerically Eq. (10b). Considering the apparent movement on the surface of the Earth, we derive an estimate of the thrust time and solve Eq. (10b) by dichotomy. This procedure yields a very fast algorithm.

Once the angle γ is known, the direction of the thrust can be determined. We can then calculate the rotation that orients the vehicle correctly for the necessary thrust vector, as described in the general algorithm (30–32).

2. *Determination of the Control for the Inner Problem*

The positions of the thrusters are supposed to be such that the thrust is along the axis u_3. There is a pair of thrusters symmetrically placed with respect to the plane (u_1, u_3) that gives a torque along u_2, and another pair of thrusters symmetrically placed with respect to the plane (u_2, u_3) that gives a torque along u_1. These thrusters can also provide a thrust (see Fig. 1 for a typical architecture).

The torque is denoted by

$$C = QV_e(D\sin a + d\cos a)(q_1 u_1 + q_2 u_2) \tag{42}$$

With this notation the mass flow rates q_i take positive or negative values, and $|q_1| + |q_2| = 1$. Let us denote by u_{3f}, the final direction of thrust to be attained and u_{30} the initial one. One can look for the solution by applying maximum torque around the axis $u_{30} \times u_{3f}$. Therefore, the torque is parallel to a vector $\cos\theta u_1 + \sin\theta u_2$, and θ is chosen such that the rotation around $u_{30} \times u_{3f}$ is positive. Let Ψ be the angle between u_{30} and u_{3f} and T_1 the rotation time:

$$T_1 = 2\sqrt{\psi I/|C|} \tag{43}$$

The mass flow rates q_1, q_2 are constant in the interval $[0, T_1/2]$ and determined by the system of equations

$$|q_1| + |q_2| = 1, \qquad \sin\theta q_1 - \cos\theta q_2 = 0 \tag{44}$$

In the interval $[T_1/2, T_1]$ the values of the mass flow-rates are $-q_1, -q_2$. We iterate the algorithm until the target on the Earth is attained up to a given precision.

C. Numerical Treatment

We applied the method of Hargraves and Paris.[6] To reduce the problem to a discrete nonlinear programming problem, the state variables were approximated by piecewise cubic polynomials and the control variables by piecewise constant functions. Recalling the general features of the method, we search for the solution of the following problem:

$$\text{minimize}\, F[x(0), x(1)] \tag{45a}$$

with the constraints

$$\frac{dx}{dt} = f(x, u, t) \tag{45b}$$

$$\phi[x(0), x(1)] \geq 0 \tag{45c}$$

$$G(x, u, t) \geq 0 \tag{45d}$$

where x, u are functions of $t \in [0, 1]$. Let $\{0 = t_1 < t_2 < \cdots < t_i < \cdots < t_m = 1\}$ be a subdivision of the interval $[0, 1]$, and $x_i = x(t_i), u_i = u_i(t)$. Between each interval $[t_i, t_{i+1}]$ the state vector is approximated by a cubic polynomial

$$y_i(t) = a_i + (t - t_i)b_i + (t - t_i)^2 c_i + (t - t_i)^3 d_i \tag{46a}$$

with

$$y_i(t_i) = x_i, \qquad y_i(t_{i+1}) = x_{i+1}, \qquad \dot{y}_i(t_i) = \dot{x}_i, \qquad \dot{y}_i(t_{i+1}) = \dot{x}_{i+1} \tag{46b}$$

and

$$\dot{x}_i = f(x_i, u_i, t_i) \tag{46c}$$

The coefficients are then determined by

$$a_i = x_i \tag{47a}$$

$$b_i = \dot{x}_i \tag{47b}$$

$$c_i = -(3/h_i^2)x_i - (2/h_i)\dot{x}_i + (3/h_i^2)x_{i+1} - (1/h_i)\dot{x}_{i+1} \tag{47c}$$

$$d_i = (2/h_i^3)x_i + (1/h_i^2)\dot{x}_i - (2/h_i^3)x_{i+1} + (1/h_i^2)\dot{x}_{i+1} \tag{47d}$$

with $h_i = t_{i+1} - t_i$.

As noted earlier the control is in the class of piecewise constant functions,

$$u(t) = u_i \qquad \text{in} \qquad [t_i, t_{i+1}] \tag{47e}$$

The collocation points are chosen at the center $t_i^c = [(t_i + t_{i+1})/2]$ of each interval. At this point

$$x_i^c \equiv y_i(t_i^c) = \tfrac{1}{2}[x_{i+1} + x_i + (h_i/4)(\dot{x}_i + \dot{x}_{i+1})] \tag{48a}$$

$$\dot{x}_i^c \equiv \dot{y}_i(t_i^c) = (3/2h_i)(x_{i+1} - x_i) - \tfrac{1}{4}(\dot{x}_i + \dot{x}_{i+1}) \tag{48b}$$

with collocation error in each interval given by

$$K_i(x_i, u_i, x_{i+1}) = \dot{x}_i^c - f(x_i^c, u_i, t_i^c) \tag{48c}$$

and the constraints (47c) and (47d) are enforced at each node, the resulting optimization problem can be stated in the following way.

Let the vector of variables be: $p^T = (x_1^T, u_1^T, x_2^T, u_2^T, \ldots, x_m^T, u_m^T)$, all of the constraints are grouped in the vectors

$$C_1^T = [K_1^T(x_1, u_1, x_2), \ldots, K_{m-1}^T, (x_{m-1}, u_{m-1}, x_m)]$$

and

$$C_2^T = [g^T(x_1, u_1), \ldots, g^T(x_m, u_m), \phi^T(x_1, x_m)]$$

Therefore, we must

$$\text{minimize}\, F(p) \qquad \text{with} \qquad C_1 = 0 \qquad \text{and} \qquad C_2 \geq 0 \tag{49}$$

For solving Eq. (49) we used a sequential quadratic programming (SQP) method, based on the package NPSOL.[7]

Others types of discretizations have been tested: cubic piecewise polynomials for state vector and linear piecewise for control, or linear piecewise for state vector and constant piecewise for control: only the discretization corresponding to best results are reported here. The choice for the distribution of points in the discretization will be discussed subsequently.

IV. Results

We have compared the analytical method and two variations in the numerical treatment: in the first, called Num1, we treat the full optimal control problem, in the second, called Num2, we break up the problem into two parts, as indicated by the asymptotic analysis: an initial rotation phase followed by a movement with constant thrust. Num2 is, of course, suboptimal.

These two methods have been tested on several cases. In the first case, the majority of the targets are such that the conditions for the validity of the asymptotic expansion are fulfilled. There are eight possible targets disposed on a rectangle of 1000 km in length and 100 km in width; the first target, denoted by impact point, is the point to be attained by the vehicle if it remains on the ballistic trajectory starting from its initial conditions, that is, without any thrust (see Fig. 3). Of course, the nearest targets are the ones for which the asymptotic model is less relevant. The ratio L/l is 50 in Fig. 3, ε is less than 10^{-1} for targets 5 and after. For each target let us compare the total time of the mission, i.e., the cost function, for the three computations, Num1, Num2, and asymptotic analysis in Table 1.

The computations have been made on a IBM-RISC 6000 work station, the computation times are given for Num1 and Num2 but not for the analytical method, as this is always less than 1 s. The agreement between the analytical and numerical method is very good in all cases. Num1 had difficulties in converging when we used a grid with equal steps and when we started from a bad initial point. Taking into account the results of the analytical analysis, we distributed the grid points. Several grids have been tested, the best results were obtained with the distribution shown in Fig. 4. The size of the grid is $h = \min\{0.0001, O(1/N^4)\}$ in the neighborhood of 0 and a distribution with the form $h = O(1/N)$ elsewhere. On the other hand, a satisfactory solution for Num2 was obtained with just a uniform distribution of grid points for both the rotation and thrust phases of the movement.

To verify both hypotheses of boundary layer and of constant thrust, we have plotted the angle with respect to time, denoted in Fig. 5 by alpha, between the direction of the vehicle (direction of thrust) and the initial direction. The dotted line corresponds to the analytical computation and the plain line corresponds to Num1. The analytical method gives a very sharp boundary layer in all cases. For the four last targets the agreement between analytical and numerical methods is very good for the size of the boundary layer and for the variations of the angle α. For example, targets 2 and 5 may seem very similar but, if we consider the relative velocity of the vehicle, the latter must apply thrust in a direction virtually perpendicular

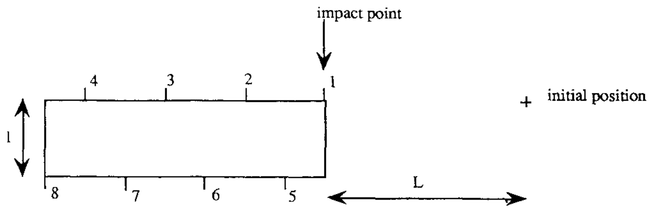

Fig. 3 Long mission.

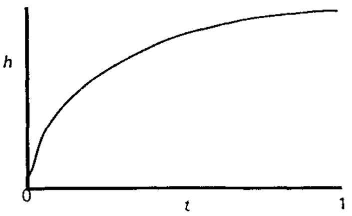

Fig. 4 Steps size of the best discretization with respect to time.

Table 1 Optimal time obtained for the mission of Fig. 3

Target	Time	Num1	Sing pert (analytic)	Num2
2	Opt., s	37.33	37.65	37.36
	Comp., min	15	—	7
3	Opt., s	75.78	76.18	75.87
	Comp., min	20	—	51
4	Opt., s	113.96	114.43	114.08
	Comp., min	33	—	8
5	Opt., s	37.48	38.02	37.34
	Comp., min	21	—	226
6	Opt., s	65.07	65.14	65.12
	Comp., min	32	—	130
7	Opt., s	100.05	100.82	100.18
	Comp., min	51	—	120
8	Opt., s	136.34	137.09	136.51
	Comp., min	85	—	188

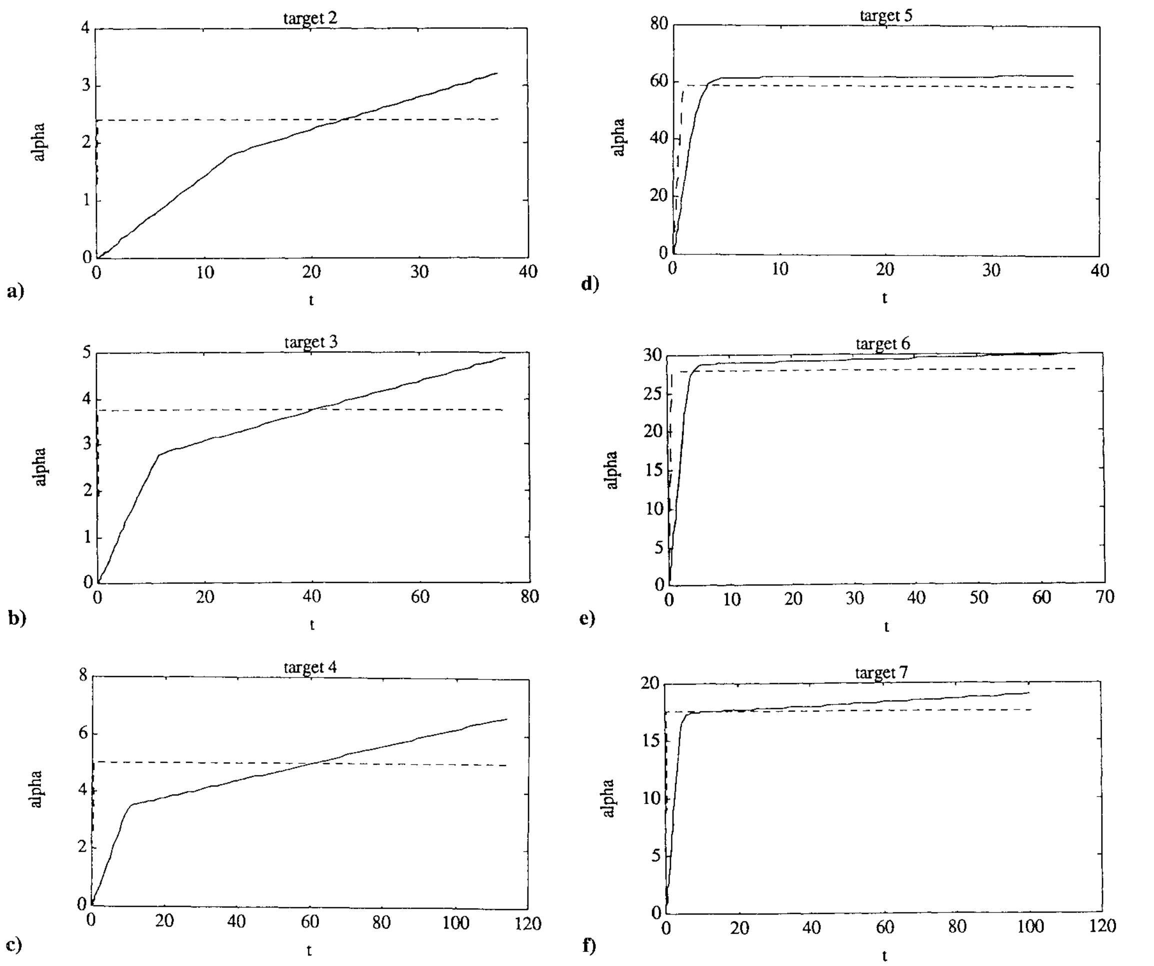

Fig. 5 Angle α with respect to time, for some Fig. 3 targets.

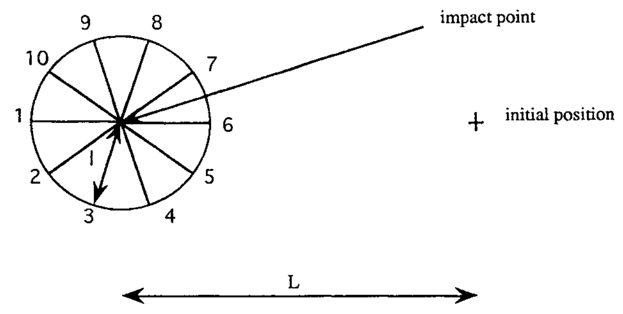

Fig. 6 Short mission.

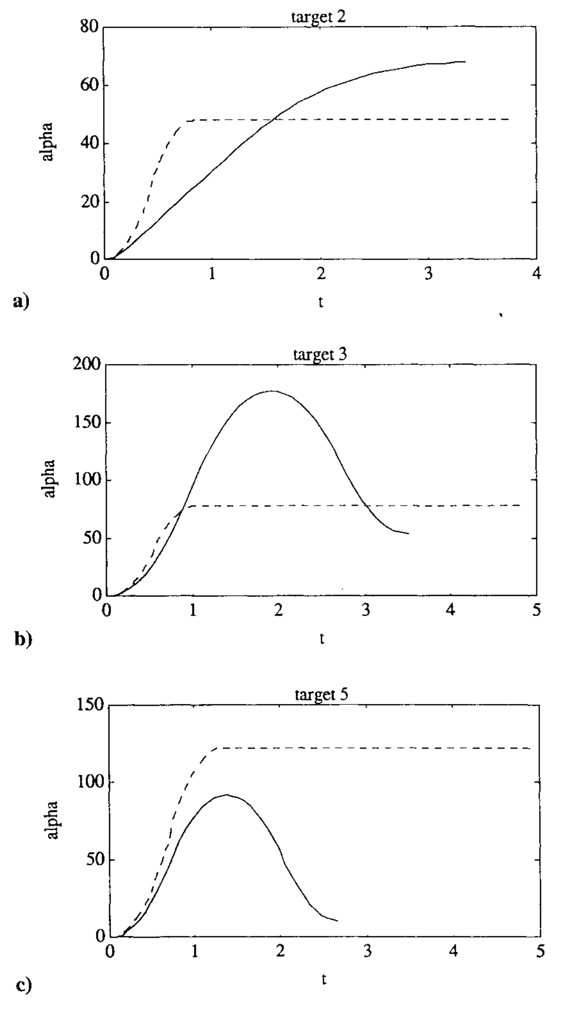

Fig. 7 Total angular variation for some targets of the Fig. 6 mission; dotted lines correspond to analytical algorithm, solid lines to the numerically computed solution.

to its initial trajectory to modify its direction sufficiently to attain target 5. A change is also necessary for target 2, but the change is much smaller as it is already in line with the initial direction. Even when the variations of the state variables do not agree, the calculated times are very similar. This is not unusual in optimal control problems, as a small variation in the cost functional can provide important variations in the control. Let us note, as it is not evident on the figures, that the angular velocity is indeed zero at the right boundary.

To examine the ability of the asymptotic model, we have tested the algorithm on a second series of targets distributed on a circle, as indicated in Fig. 6, the radius l of the circle is about 10 km. The ratio L/l is 300 in Fig. 6, the parameter ε is of order 1 for all of the targets in this set. The asymptotic assumptions are, therefore, no longer valid, and so we expect there to be large differences between the numerical and asymptotic schemes. The results are reported

Table 2 Optimal computed time obtained numerically and analytically for targets 1–5 of the mission of Fig. 6

	Numerical results	Analytical results	
Target	Optimal time, s	Optimal time, s	Total angular variation, deg
1	2.067	2.544	5.719
2	3.348	3.793	47.834
3	3.514	4.818	70.200
4	3.305	5.518	99.774
5	2.669	4.897	127.341

Table 2, since the targets are symmetrically dispersed we report only the results for the targets 1–5.

We have reported some results in Fig. 7. The total angular variation is plotted: the dotted line corresponds to the analytical algorithm and the solid line to the numerically computed solution. Although no problems of convergence occurred, the numerical computations revealed many local minima for each target, and the global minimum was difficult to determine. Indeed, some of the minima found had an optimal time that was greater than the one obtained analytically. Hence, these results show that if the asymptotic assumptions are no longer valid, as is the case here, the method nevertheless remains robust and gives a control that satisfies all of the constraints. The mission time, of course, is not quite as close to the optimal time. Since these mission times are so short, however, the difference between the two is of little practical importance.

V. Conclusion

We have seen in this paper that a singular perturbation technique can be applied to the following optimal control problem: Find the thrust (i.e., the control) that must be apply to a vehicle during an extra-atmospheric movement such that it reaches in a minimum time a point where it must release an object that must attain a given target on the surface of the Earth. The asymptotic analysis yields a search for controls that present a short initial rotational phase followed by a constant thrust. Therefore, the Euler angles that describe the attitude of the vehicle around its center of mass present very fast variations near the initial time; that is to say, there exists a so-called rotational boundary layer at the origin. We have then derived a very fast semianalytical computation that fits quite well with the results of a general purpose optimization code. The differences in the computation time between Num1 (i.e., the full optimal control problem) and Num2 (i.e., a rotation phase followed by constant thrust) come from the fact that Num2 has been initialized with less care than the complete optimization Num1. Nevertheless, the pure numerical treatments Num1 and Num2 took advantage of the asymptotic analysis and of the analytical results through an adaptation of the mesh that anticipates the presence of an initial boundary layer. So far we have imposed no condition on the vehicle attitude at the launching. If terminal conditions were added (by fixing the final values of the Euler angles), then a rotational boundary layer would also appear near the launching point. This addition has been made with success. Since the analysis is similar to that already presented, however, we have omitted its description in this paper for the sake of simplicity and brevity.

This analysis could be also applied to other problems, such as orbital transfer, with the same success. So far we have optimized the mission time for a given fixed architecture. If we were to consider the optimal mission time as a function of the architecture, we could use the algorithm described to improve the latter in an attempt to further reduce the optimal time. Such a process would require many calculations of the cost function (the optimal time). In this case the calculation speed becomes important, which is why this method has been used with success in an optimum design problem.[8]

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